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On $N=4$ super Riemann surfaces and the superconformal semigroup

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Abstract. The finite $N=4$ superconformal (SCf) transformations are obtained and classified into three types by means of the permanent of specific matrices. The properties of the latter are studied in connection with complex plane non-Euclidean geometry. The explicit expression for the $N=4$ SCf transformations allows them to be used as the transition functions for $N=4$ split and non-split super Riemann surfaces and SCf embeddings. The existence of the semigroup of SCf transformations is indicated.

1. Introduction

$N=4$ superconformal (SCf) symmetry in two dimensions was first employed to investigate the colour SU(2) string [1], but its covariant formulation in terms of the $N=4$ σ-model was added only 10 years later [2]. Further, when building realistic models by superstring compactification [3], a relation between the $N=4$ world-sheet SCf symmetry and the $N=2$ spacetime supersymmetry was established [4]. Moreover, the hidden $N=4$ global SCf symmetry on the world-sheet was traced back [5] even for the U(1) string [6]. Recently, much progress has been made in studying $N=4$ SCf algebras [7] and their representation theory [8]. The $N=4$ (rigid and local) supersymmetry in two-dimensional σ-models is the largest possible one allowed by chiral invariance argument [9] and holonomy group considerations [10]. Also, the anomaly term in the stress-energy tensor SCf transformation corresponding to the SCf algebra central term exists only for $N\leqslant 4$ [11]. On the other hand the calculation of fermionic string amplitudes in the Polyakov approach [12] comes to the finite-dimensional integration over a supermoduli space [13] counting off the inequivalent super Riemann surfaces (SRSs) [14]. Following the local definition [15] the $N=4$ SRSs can be viewed as a collection of superdomains glued by $N=4$ SCf transformations [16] (we study this from the algebraic viewpoint here and do not lay stress on the consistent $N=4$ string formulation which meets difficulties with ghosts, critical dimensions, etc. [1]).

In this paper a detailed analysis of $N=4$ SCf symmetry peculiarities is carried out. An exact shape of the general finite $N=4$ SCf transformations corresponding to various types of $N=4$ SRSs and playing some part in the above-mentioned investigations is obtained. Almost all results can be applied to lower Ns.

In section 2 we rewrite the $N=4$ SCf conditions on a complex basis using the permanent of some matrices (we call them scf matrices) having specific properties which are examined in section 3. The new symmetry which is generated by the fractional linear transformations associated with scf matrices on the complex plane is found in

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section 4 where suitable analogues of the cross ratio and non-Euclidean distance are presented. In section 5 we express the $N=4$ Berezinian via the permanent and outline SCf transformation of $N=4$ superdifferentials. The soul-extended analogue of the split $N=4$ SRS transition functions is introduced in section 6. Weakening of invertibility results in the transformations which form a semigroup called an $N=4$ SCf semigroup. In section 7 we consider the transition functions for the non-split $N=4$ SRSs, fractional linear transformations and the subsemigroup of transformations containing no even functions at all. Some generalities about the $N=4$ SCf semigroup are discussed in section 8.

2. $N=4$ SCf conditions

The complex (1|4)-dimensional superspace [1, 2] can be parametrized by one even (z) and four odd ($\theta_1-\theta_4$) coordinates (we consider only a holomorphic sector). On a complex basis ($\theta_1^\pm=(\theta_1\pm \mathrm{i}\theta_2)/\sqrt{2}$, $\theta_2^\pm=(\theta_3\pm \mathrm{i}\theta_4)/\sqrt{2}$) the left superderivatives $D_i^\pm=\partial/\partial\theta_i^\mp+\theta_i^\pm\partial/\partial z$ form the algebra $\{D_i^+, D_j^-\}=2\delta_{ij}\partial/\partial z$ ($i,j=1,2$), other anticommutators vanishing. Under superanalytic transformation $Z\to\tilde{Z}(Z)$ which has no $\bar{Z}$-dependence (see [17] for the rigorous definitions), where $Z=(z,\theta_i^+,\theta_i^-)\in\mathbb{C}^{1,4}$ the superderivatives transform as

$$D_i^\pm=(D_i^\pm\tilde{\theta}_j^-)\tilde{D}_j^++(D_i^\pm\tilde{\theta}_j^+)\tilde{D}_j^-+(D_i^\pm\tilde{z}-(D_i^\pm\tilde{\theta}_j^-)\tilde{\theta}_j^+-(D_i^\pm\tilde{\theta}_j^+)\tilde{\theta}_j^-)\partial/\partial\tilde{z}. \tag{1}$$

A superanalytic transformation is superconformal if the inhomogeneous term in (1) becomes zero and superderivatives $D_i^\pm$ transform covariantly [11, 14–16]. Then, an $N=4$ SRS can be defined locally as the (1|4)-dimensional complex supermanifold [17] patched from maps by means of the $N=4$ SCf transformation as the transition function on overlapping maps [15, 16].

It follows from (1) that the $N=4$ SCf condition is

$$D_i^\pm\tilde{z}=(D_i^\pm\tilde{\theta}_j^-)\tilde{\theta}_j^++(D_i^\pm\tilde{\theta}_j^+)\tilde{\theta}_j^-. \tag{2}$$

Manipulating superderivative algebra we derive

$$\partial\tilde{z}/\partial z+\tilde{\theta}_i^+\partial\tilde{\theta}_i^-/\partial z+\tilde{\theta}_i^-\partial\tilde{\theta}_i^+/\partial z=\operatorname{per}H_{11}+\operatorname{per}H_{21}=\operatorname{per}H_{12}+\operatorname{per}H_{22} \tag{3}$$

$$\operatorname{scf}_i H_{11}+\operatorname{scf}_i H_{21}=\operatorname{scf}_i H_{12}+\operatorname{scf}_i H_{22}=0 \tag{4}$$

$$H_{11}^{\mathrm{T}}H_{12}^{\mathrm{M}}+H_{21}^{\mathrm{T}}H_{22}^{\mathrm{M}}=0 \tag{5}$$

where

$$H_{ij}=\begin{pmatrix} D_j^+\tilde{\theta}_i^- & D_j^-\tilde{\theta}_i^- \\ D_j^+\tilde{\theta}_i^+ & D_j^-\tilde{\theta}_i^+ \end{pmatrix} \tag{6}$$

H_{ij}^{T} is the transposed matrix and H_{ij}^{M} is the matrix of minors. The permanent [18] is defined as

$$\operatorname{per}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=ad+bc \tag{7}$$

and we introduce a useful matrix function $\operatorname{scf}_i A$ by

$$\operatorname{scf}_1\begin{pmatrix} a & b \\ c & d \end{pmatrix}=ac \qquad \operatorname{scf}_2\begin{pmatrix} a & b \\ c & d \end{pmatrix}=bd. \tag{8}$$

Thus it is the specific 4×4 block matrix $H=(H_{ij})$ that fully determines the shape of $N=4$ scf transformation and controls the behaviour of various objects on $N=4$ SRSs [16, 19]. Note that most parts of the above relations, and those below, are also valid for $N=2$. In this case after discarding, for example, quantities with the subscript 2 we deal with the special even 2×2 matrix having a fixed permanent and satisfying (4), which becomes $\mathrm{scf}_i H_{11}=0$. We call this property an $N=2$ scf property and such a 2×2 matrix an $N=2$ scf matrix. Going to the general 4×4 block matrix $H=(H_{ij})$ we call it an $N=4$ scf matrix, if the blocks H_{ij} satisfy (3)-(5). It is reasonable to dwell on their basic features.

3. Permanent and scf matrices

Let A be a 2×2 matrix with entries from Λ_0 being the even part of a complex Grassmann algebra [20]. Defining a scalar product, usually $A\times B=\mathrm{tr}\,AB^{\mathrm{T}}$, one has

$$\mathrm{per}(A+B)=\mathrm{per}\,A+\mathrm{per}\,B+A\times B^{\mathrm{M}} \tag{9}$$

and, in particular,

$$\begin{aligned}&\mathrm{per}(A-\kappa I)=\kappa^2-\kappa\,\mathrm{tr}\,A+\mathrm{per}\,A\\&\mathrm{per}(A-A^{\mathrm{MT}})=2\,\mathrm{per}\,A-\mathrm{tr}\,A^2.\end{aligned} \tag{10}$$

So that the permanent can be determined as follows:

$$\mathrm{per}\,A=\tfrac{1}{2}A\times A^{\mathrm{M}}. \tag{11}$$

This results also from the remarkable relation

$$A^{\mathrm{MT}}A=\begin{pmatrix}\mathrm{per}\,A & 2\,\mathrm{scf}_2\,A\\ 2\,\mathrm{scf}_1\,A & \mathrm{per}\,A\end{pmatrix} \tag{12}$$

explaining why the $N=2$ scf matrix A_{scf} defined by $\mathrm{scf}_i\,A_{\mathrm{scf}}=0$ is so marked (we omit '$N=2$' in evident cases). It is worth comparing (12) with the usual $A^{\mathrm{DT}}A=I\det A$, A^{D} being a matrix of cofactors and I being the unit matrix. Hence, only for A_{scf} do many relations become symmetric under $\mathrm{per}\,A_{\mathrm{scf}}\leftrightarrow\det A_{\mathrm{scf}}$ and $A^{\mathrm{M}}_{\mathrm{scf}}\leftrightarrow A^{\mathrm{D}}_{\mathrm{scf}}$. For example, a per inverse matrix

$$A_{\mathrm{scf}}^{-1,\mathrm{per}}=\frac{A_{\mathrm{scf}}^{\mathrm{MT}}}{\mathrm{per}\,A_{\mathrm{scf}}} \tag{13}$$

can be defined if $\varepsilon(\mathrm{per}\,A_{\mathrm{scf}})\neq0$, where ε is the body map [17] (every element of Λ_0 has the number part (body) and the nilpotent part (soul), so the body map discards the latter). If A_{scf} is diagonal or off-diagonal then $A_{\mathrm{scf}}^{-1,\mathrm{per}}=A_{\mathrm{scf}}^{-1}$. Furthermore,

$$\mathrm{tr}\,A^n_{\mathrm{scf}}=a^n+d^n+[1+(-1)^n](bc)^{n/2} \tag{14}$$

and

$$\begin{pmatrix}\mathrm{per}\\ \mathrm{det}\end{pmatrix}^n A_{\mathrm{scf}}=\begin{pmatrix}\mathrm{per}\\ \mathrm{det}\end{pmatrix}A^n_{\mathrm{scf}}=(ad)^n+(\pm1)^n(bc)^n. \tag{15}$$

So the bodies of the permanent and determinant of the scf matrix can only differ by sign or vanish; as for their souls, (15) is the system of equations. Here the Binet-Cauchy formula for permanents [18] reduces to the same form as for determinants:

$$\mathrm{per}(A_{\mathrm{scf}}B_{\mathrm{scf}})=\mathrm{per}\,A_{\mathrm{scf}}\,\mathrm{per}\,B_{\mathrm{scf}} \tag{16}$$

and

$$\begin{aligned} &\operatorname{per} A_{scf}^{-1} = \operatorname{per}^{-1} A_{scf} \\ &\operatorname{per}(A_{scf} B_{scf} A_{scf}^{-1}) = \operatorname{per} B_{scf}. \end{aligned} \tag{17}$$

But the connection

$$(2 \operatorname{per} A_{scf} - \operatorname{tr} A_{scf}^2)(2 \operatorname{per} A_{scf} + \operatorname{tr} A_{scf}^2 - \operatorname{tr}^2 A_{scf}) = 0 \tag{18}$$

between the permanent and the trace comes into existence only for the scf matrix and holds as a whole without vanishing each of the factors. It reflects the fact that A_{scf} entries form the ring containing divisors of zero and proper ideals [17]. One can check that the scf property is preserved under usual multiplication. Therefore the scf matrices form a linear semigroup [21] to say the least. We call A_{scf} a body scf matrix if $\varepsilon(\operatorname{per} A_{scf}) \neq 0$ and a soul one if $\varepsilon(\operatorname{per} A_{scf}) = 0$, but A_{scf} having the vanishing permanent is called a zero scf matrix. The body scf matrices form the linear group, being a subgroup of $GL(2, \Lambda_0)$ [20]. The scf property of the soul scf matrix is realized by virtue of the existence of divisors of zero among A_{scf} entries. The soul 2×2 scf matrices appear when studying the semigroup of $N = 2$ scr transformations [22].

Consider the scf property consequences in changing the complex basis to a coordinate one. Let $\tilde{A} = U^{-1} A U$, where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

Then

$$\tilde{A}^T \tilde{B} = U^{-1} A^T B^M U \tag{19}$$

and the main relation between the orthogonality on a coordinate basis and the scf property in the complex one is (see (12))

$$\tilde{A}^T \tilde{A} = \operatorname{per} A I + \operatorname{scf}_1 A(\sigma_3 + i\sigma_1) + \operatorname{scf}_2 A(\sigma_3 - i\sigma_1) \tag{20}$$

σ_j being the Pauli matrices. It is easily seen that the body scf matrix A_{scf} after normalization by $\sqrt{\operatorname{per} A_{scf}}$ becomes similar to the $O(2, \Lambda_0)$ matrix [20]. Such a trick is impossible for A_{scf} being the soul scf matrix or zero one.

Let $B = (A_{ij})$ be a 4×4 block matrix with entries from Λ_0 and $\tilde{B} = (\tilde{A}_{ij}) = (U^{-1} A U)$ be the similar one on a coordinate basis. Using (19) and (20) we obtain

$$\begin{aligned} &\tilde{B}^T \tilde{B} = \begin{pmatrix} P_1 & Q \\ Q & P_2 \end{pmatrix} \\ &P_j = (\operatorname{per} A_{1j} + \operatorname{per} A_{2j}) I + (\operatorname{scf}_1 A_{1j} + \operatorname{scf}_1 A_{2j})(\sigma_3 + i\sigma_1) \\ &\qquad + (\operatorname{scf}_2 A_{1j} + \operatorname{scf}_2 A_{2j})(\sigma_3 - i\sigma_1) \\ &Q = U^{-1}(A_{11}^T A_{12}^M + A_{21}^T A_{22}^M) U. \end{aligned} \tag{21}$$

Thus the $N = 4$ *scf property resulting in* $\tilde{B}^T \tilde{B} = RI$ takes the form

$$\begin{aligned} &\operatorname{per} A_{11} + \operatorname{per} A_{21} = \operatorname{per} A_{12} + \operatorname{per} A_{22} = R \\ &\operatorname{scf}_i A_{11} + \operatorname{scf}_i A_{21} = \operatorname{scf}_i A_{12} + \operatorname{scf}_i A_{22} = 0 \\ &A_{11}^T A_{12}^M + A_{21}^T A_{22}^M = 0 \end{aligned} \tag{22}$$

(cf (3)–(5)). We call $B_{\rm scf}$ satisfying (22) an $N=4$ body, soul or zero scf matrix according to $\varepsilon(R)\neq 0$, $\varepsilon(R)=0$ or $R=0$. The body $B_{\rm scf}$ normalized by $\sqrt{R}$ is similar to $\tilde{B}_n \in \mathrm{O}(4, \Lambda_0)$ [20]. To derive the determinant of the $N=4$ scf matrix we use the relation

$$\det(A_1^{\rm MT}A_1 + A_2^{\rm MT}A_2) = (\operatorname{per} A_1 + \operatorname{per} A_2)^2 \tag{23}$$

which holds for any A_j satisfying $\mathrm{scf}_i\, A_1 + \mathrm{scf}_i\, A_2 = 0$. Then

$$\det B_{\rm scf} = \kappa R^2 \tag{24}$$

and after normalization $\kappa = 1$ leads to $\tilde{B}_n \in \mathrm{SO}(4, \Lambda_0)$ and $\kappa = -1$ results in $\tilde{B}_n$ having the general $\mathrm{O}(4, \Lambda_0)$ symmetry (for the body $B_{\rm scf}$). The other types of $B_{\rm scf}$ do not allow such normalization. If some $\varepsilon(\det A_{ij}) = 0$ then we shall use a suitable relation from the general ones:

$$\begin{aligned}\det B_{\rm scf} &= R^2 \frac{\det A_{11}}{\det A_{22}} = R^2 \frac{\det A_{22}}{\det A_{11}} \\ &= -R^2 \frac{\det A_{12}}{\det A_{21}} = -R^2 \frac{\det A_{21}}{\det A_{12}}.\end{aligned} \tag{25}$$

Note also that the permanent of the body $B_{\rm scf}$ has the form

$$\operatorname{per} B_{\rm scf} = (\operatorname{per} A_{11} - \operatorname{per} A_{21})(\operatorname{per} A_{22} - \operatorname{per} A_{12}). \tag{26}$$

The above relations are very useful and somewhat indispensable for the first-hand analysis of $N=4$ SCF symmetry manifestations on a complex basis.

4. Non-Euclidean plane and scf matrices

Here we give attention to a fascinating niche for scf matrices. As is well known, any $A_{\rm scf}$ can represent a fractional linear transformation of $z \in \mathbb{C}^{1,0}$:

$$f(z) = \frac{az+b}{cz+d} \tag{27}$$

which is called a per mapping if $ac = bd = 0$. Since the scf property is preserved under matrix multiplication, whole per mappings generally form a semigroup. We first note that the $N=2$ scf property ($ac = bd = 0$) coincides with some of the conditions for the per mapping to commute with a normal one [23]. Moreover, it is easily seen that a soul fixed point having $\varepsilon(z_{\rm soul}) = \varepsilon(b) = 0$ appears here as

$$z_{\rm soul} = \frac{b}{d-a}. \tag{28}$$

In addition to the conventional studies of $f(z)$ [23, 24] we can observe the following 'mirror' condition to hold for the per mapping:

$$f(z_1) + f(z_2) = \frac{\operatorname{per} A_{\rm scf}}{(cz_1+d)(cz_2+d)}(z_1+z_2) \tag{29}$$

which clarifies the terminology. For $A_{\rm scf}$ entries being real, we obtain the relations

$$\operatorname{Re} f(z) = \frac{\operatorname{per} A_{\rm scf}}{|cz+d|^2} \operatorname{Re} z \tag{30}$$

and

$$\frac{|f(z_1)+f(z_2)|^2}{\mathrm{Re}\, f(z_1)\,\mathrm{Re}\, f(z_2)}=\frac{|z_1+z_2|^2}{\mathrm{Re}\, z_1\,\mathrm{Re}\, z_2} \tag{31}$$

so that the isometric 'unit circle' can be defined by $|cz+d|=\sqrt{\mathrm{per}\, A_{\mathrm{scf}}}$. Further, it is natural to introduce a 'cross ratio' as follows:

$$D^+(z_1,z_2,z_3,z_4)=\frac{(z_1+z_3)(z_2+z_4)}{(z_1+z_4)(z_2+z_3)} \tag{32}$$

being invariant under a per mapping

$$D^+(f(z_1),f(z_2),f(z_3),f(z_4))=D^+(z_1,z_2,z_3,z_4)=r \tag{33}$$

and having the properties of the usual cross ratio [23]. The group of z_i permutations S_4^+ is homomorphic to a finite group consisting of six elements

$$\left\{r,\frac{1}{r},1+r,\frac{1}{1+r},\frac{1+r}{r},\frac{r}{1+r}\right\} \tag{34}$$

for $\varepsilon(r)\neq 0$, of four elements for $\varepsilon(r)=0$ and of three elements for $r=(-1\pm\sqrt{5})/2$ or for $r=1$ (in the latter case the points form a 'harmonic set'). The next step is to define the 'distance' from z_1 to z_2 inside the 'unit disk' as

$$d^+(z_1,z_2)=\ln D^+(z_1,z_2,z_3,z_4) \tag{35}$$

where all points lie on the same 'geodesics' determined by z_1 and z_2 only, while z_3 and z_4 are end points of that on the 'unit circle'. It can be shown that r is real if the points lie on the 'unit circle', and 'angles' between 'geodesics' are also calculable in terms of the 'cross ratio'. Using (31) we can choose the 'distance' on the 'upper half plane' now becoming the right plane ($\mathrm{Re}\, z>0$) in the following way:

$$d^+_{\mathrm{Re}}(z_1,z_2)=\mathrm{Arch}\left(1+\frac{|z_1+z_2|^2}{\mathrm{Re}\, z_1\,\mathrm{Re}\, z_2}\right). \tag{36}$$

We finally note that the per mapping satisfies

$$n_1 f(z_1)+n_2 f(z_2)=\frac{\mathrm{per}\, A_{\mathrm{scf}}(n_1+n_2)(z_1+z_2)+\det A_{\mathrm{scf}}(n_1-n_2)(z_1-z_2)}{2(cz_1+d)(cz_2+d)}. \tag{37}$$

Hence, the scf matrix represents the per mapping giving rise to the additional symmetry condition (29) on the complex plane. To find any meaning of it in line with the plane non-Euclidean geometry [23] we can surmise that the suitable 'Euclidean distance' might be $|z_1+z_2|$. Then almost all related constructions of [23, 24] could be repeated here with some evident changes.

5. $N=4$ Berezinian and superdifferentials

In the (1|4)-dimensional superspace the Berezin integration measure [25] transforms under the general superanalytic transformation $Z\to\tilde{Z}$ by means of the Berezinian $\mathrm{Ber}(\tilde{Z}/Z)$ (the superanalogue of the Jacobian), which on a complex basis can be presented as follows (for $\varepsilon(\det H)\neq 0$):

$$\mathrm{Ber}(\tilde{Z}/Z)=\frac{\partial\tilde{z}/\partial z+\tilde{\theta}_i^+\partial\tilde{\theta}_i^-/\partial z+\tilde{\theta}_i^-\partial\tilde{\theta}_i^+/\partial z-(\partial\tilde{\theta}_i^+/\partial z,\partial\tilde{\theta}_i^-/\partial z)(H^{-1})\begin{pmatrix}\Delta_i^-\\ \Delta_i^+\end{pmatrix}}{\det H} \tag{38}$$

where

$$\Delta_i^{\pm} = D_i^{\pm}\tilde{z} - (D_i^{\pm}\tilde{\theta}_j^{+})\tilde{\theta}_j^{-} - (D_i^{\pm}\tilde{\theta}_j^{-})\tilde{\theta}_j^{+} \tag{39}$$

vanishes for the SCf transformation due to (2). Further, using the SCf condition (3) and the formula (24) for the determinant of the $N = 4$ scf matrix H (being the body scf matrix here) we obtain

$$\mathrm{Ber}(\tilde{Z}/Z) = \frac{\kappa}{J} \tag{40}$$

where

$$J = \mathrm{per}\, H_{11} + \mathrm{per}\, H_{21} = \mathrm{per}\, H_{12} + \mathrm{per}\, H_{22}. \tag{41}$$

With the SCf transformation serving as the transition function for the $N = 4$ SRS, the value of κ is said to distinguish between the untwisted SRS ($\kappa = 1$) and the twisted one ($\kappa = -1$) [16]. To define line integrals properly [26] and to investigate line bundles on an SRS [27] they introduce [14, 16] the Abelian (or basic) SCf differentials $\mathrm{d}\tau_i^{\pm}$. The corresponding cotangent space is dual to the SCf-invariant (0|4)-dimensional tangent subspace spanned by $D_i^{\pm}$ only. So the Abelian SCf differentials transform inversely to superderivatives $\mathrm{d}\tilde{\tau} = \mathrm{d}\tau H$, $D = H\tilde{D}$ (in matrix notation), which guarantees the SCf invariance of the operator $\mathrm{d}\tau\, D$ used to construct fermionic string action [12]. Besides, they satisfy the dual relation $\{\mathrm{d}\tau_i^{+}, \mathrm{d}\tau_j^{-}\} = 2\delta_{ij}\, \mathrm{d}Z$, where $\mathrm{d}Z = \mathrm{d}z + \theta_i^{+}\, \mathrm{d}\theta_i^{-} + \theta_i^{-}\, \mathrm{d}\theta_i^{+}$ transforms as $\mathrm{d}\tilde{Z} = J\, \mathrm{d}Z$. This relation can be interpreted as the $N = 4$ generalization of $\mathrm{d}\tilde{z} = (\partial\tilde{z}/\partial z)\, \mathrm{d}z$ [16], and (since $\mathrm{d}Z$ corresponds to the metric on the complex $N = 4$ superplane) J is the scaling factor for it. Note that for general N one has

$$\mathrm{Ber}(\tilde{Z}/Z) = \kappa(\det H)^{(2-N)/N} \tag{42}$$

and for $N \neq 2$ one obtains the fundamental equation

$$\mathrm{d}\tilde{Z} = [\kappa\, \mathrm{Ber}(\tilde{Z}/Z)]^{2/(2-N)}\, \mathrm{d}Z \tag{43}$$

which can be viewed as another equivalent definition of the transition functions for an N-extended SRS.

To cover generalizing possibilities carrying further information about SCf symmetry it is natural to bring all of the transformations satisfying (2) in, including those which meet some slackened invertibility requirements. It is also convenient to refer to them as $N = 4$ SCf transformations forming a semigroup as a whole. They should next be classified in terms of J (41) since H is the $N = 4$ scf matrix (see section 3): the transformation satisfying (2)-(5) will be called a body, soul or zero transformation according to $\varepsilon(J) \neq 0$, $\varepsilon(J) = 0$ or $J = 0$. It is obvious that the body transformations are invertible and the $N = 4$ Berezinian (40) is well defined for them only, hence they form the subgroup of the full SCf semigroup. The soul transformations have a non-trivial structure in the soul direction and become degenerate after the body mapping; also, they can be partially invertible. For $N = 1$, 2 the soul transformations have been considered previously [22, 28].

6. The soul-extended split $N = 4$ SRSs

The split $N = 4$ SRS is a special type of Riemann surface and has a spin structure. The corresponding supermanifold is said to be a vector bundle over a Riemann surface with an even coordinate as the base coordinate and odd coordinates as the fibre ones [14, 16]. In this case the transition functions are

$$\tilde{z}=f(z)$$
$$\tilde{\theta}_i^{\pm}=g_{ij}^{\pm\mp}(z)\theta_j^{\mp}+g_{ij}^{\pm\pm}(z)\theta_j^{\mp} \qquad (44)$$

and so the SCf conditions (3)-(5) take the form

$$\operatorname{per} G_{11}+\operatorname{per} G_{21}=\operatorname{per} G_{12}+\operatorname{per} G_{22}=f'(z) \qquad (45)$$

$$\operatorname{scf}_i G_{11}+\operatorname{scf}_i G_{21}=\operatorname{scf}_i G_{12}+\operatorname{scf}_i G_{22}=0 \qquad (46)$$

$$G_{11}^{\mathrm{T}}G_{12}^{\mathrm{M}}+G_{21}^{\mathrm{T}}G_{22}^{\mathrm{M}}=0 \qquad (47)$$

$$\begin{aligned} G_{11}^{\mathrm{T}}G_{11}^{\mathrm{M}\prime}+G_{21}^{\mathrm{T}}G_{21}^{\mathrm{M}\prime} &= G_{11}^{\mathrm{T}\prime}G_{11}^{\mathrm{M}}+G_{21}^{\mathrm{T}\prime}G_{21}^{\mathrm{M}} \\ &= G_{12}^{\mathrm{T}}G_{12}^{\mathrm{M}\prime}+G_{22}^{\mathrm{T}}G_{22}^{\mathrm{M}\prime} \\ &= G_{12}^{\mathrm{T}\prime}G_{12}^{\mathrm{M}}+G_{22}^{\mathrm{T}\prime}G_{22}^{\mathrm{M}} \end{aligned} \qquad (48)$$

where

$$G_{ij}=\begin{pmatrix} g_{ij}^{+-}(z) & g_{ij}^{++}(z) \\ g_{ij}^{--}(z) & g_{ij}^{-+}(z) \end{pmatrix} \qquad (49)$$

(for the infinitesimal form of (44) see [1]). Now we assume the component functions from (44) to contain odd parameters and to have souls in general, while the shape of the transformations and the SCf conditions remain the same as for the split case (44)-(48). This results in $G=(G_{ij})$ being the $N=4$ scf matrix (see section 3), here satisfying the strong additional restriction (48). The body $N=4$ scf matrix G after normalization by $\sqrt{f'(z)}$ becomes similar to $\tilde{G}_n\in \mathrm{O}(4,\Lambda_0)$, obeying $\det \tilde{G}_n=k$, which gives the above-mentioned kinds of soul-extended split SRSs. For them the Berezinian (40) reads

$$\operatorname{Ber}(\tilde{Z}/Z)=\frac{k}{f'(z)}. \qquad (50)$$

Consider the structure of possible solutions of (45)-(48) in some detail. First suppose the SCf transformation (44) to be the body transformations and the blocks G_{ij} to be the body $N=2$ scf matrices (see (8) and below). Then we obtain

$$\begin{aligned} \tilde{z}&=f(z) \\ \tilde{\theta}_1^{\pm}&=g_{\pm}(z)\hat{\theta}_1^{\mp}+h_{\pm}(z)\hat{\theta}_2^{\pm} \\ \tilde{\theta}_2^{\pm}&=-h_{\mp}(z)\hat{\theta}_1^{\pm}+g_{\mp}(z)\hat{\theta}_2^{\mp} \end{aligned} \qquad (51)$$

the extra conditions

$$g_+(z)g_-(z)+h_+(z)h_-(z)=f'(z) \qquad (52)$$

and

$$g_+(z)g'_-(z)+h'_+(z)h_-(z)=g'_+(z)g_-(z)+h_+(z)h'_-(z) \qquad (53)$$

following from (45) and (48), where $\hat{\theta}_i^{\pm}=\theta_i^{\pm}\exp(\pm q_i)$ are the global rotations in the odd sector. Hereafter we list one solution from the evident corresponding set only.

The transformations having $h_+(z)=0$, $h_-(z)=0$ and $h_+(z)=h_-(z)=0$ form the separate subgroups of the scf semigroup. In the other situation the local rotations

$$\begin{aligned} g_\pm(z) &= g(z)\,\mathrm{e}^{\pm q(z)} \\ h_\pm(z) &= h(z)\,\mathrm{e}^{\pm p(z)} \end{aligned} \tag{54}$$

appear, the entries satisfying

$$\begin{aligned} g^2(z)+h^2(z) &= f'(z) \\ q'(z)g^2(z) &= p'(z)h^2(z). \end{aligned} \tag{55}$$

The most symmetric solutions of (54) and (55) are $g(z)=h(z)=\sqrt{f'(z)/2}$ and $q(z)=p(z)$. It follows from (55) that for $h(z)$ being pure soul and nilpotent, $q(z)$ becomes constant while $p(z)$ is not fixed, implying both a local rotation and a global one. Recall that for $h(z)=0$ local rotations do not arise at all.

In case two of G_{ij} are the soul $N=2$ scf matrices (i.e. having pure soul permanent, see (7) and (8)) and we can derive

$$\begin{aligned} \tilde{z} &= f(z) \\ \tilde{\theta}_1^\pm &= \hat{\theta}_1^\pm\sqrt{f'(z)}+(\hat{\theta}_2^\pm\alpha_1^\mp(z)+\hat{\theta}_2^\mp\alpha_1^\pm(z))\alpha_2^\pm(z) \\ \tilde{\theta}_2^\pm &= \hat{\theta}_2^\pm\sqrt{f'(z)}+(\hat{\theta}_1^\pm\alpha_2^\mp(z)+\hat{\theta}_1^\mp\alpha_2^\pm(z))\alpha_1^\pm(z) \end{aligned} \tag{56}$$

with the extra conditions

$$\begin{aligned} (\alpha_1^+(z)\alpha_1^{-\prime}(z)+\alpha_1^-(z)\alpha_1^{+\prime}(z))\alpha_2^+(z)\alpha_2^-(z) &= 0 \\ (\alpha_2^+(z)\alpha_2^{-\prime}(z)+\alpha_2^-(z)\alpha_2^{+\prime}(z))\alpha_1^+(z)\alpha_1^-(z) &= 0 \end{aligned} \tag{57}$$

where Roman and Greek letters are used to denote even functions $\mathbb{C}^{1,0}\to\mathbb{C}^{1,0}$ and odd ones $\mathbb{C}^{1,0}\to\mathbb{C}^{0,1}$ respectively. The possible solutions of (57) are $\alpha_1^\mp(z)=\alpha_2^\mp(z)$ and $\alpha_i^\mp(z)=\beta_i^\mp\exp(t_iz)$.

At the end of the section we give attention to the soul $N=4$ scf transformation (being a global one in the odd sector for simplicity) which has the form

$$\begin{aligned} \tilde{z} &= 4\alpha_1^+\alpha_1^-\alpha_2^+\alpha_2^- z+c \\ \tilde{\theta}_1^\pm &= (\pm\hat{\theta}_1^\pm\alpha_2^\mp+\hat{\theta}_1^\mp\alpha_2^\pm)\alpha_1^\pm+(\pm\hat{\theta}_2^\pm\alpha_2^\mp+\hat{\theta}_2^\mp\alpha_2^\pm)\alpha_1^\mp \\ \tilde{\theta}_2^\pm &= (\pm\hat{\theta}_1^\pm\alpha_1^\mp+\hat{\theta}_1^\mp\alpha_1^\pm)\alpha_2^\pm+(\pm\hat{\theta}_2^\pm\alpha_1^\mp+\hat{\theta}_2^\mp\alpha_1^\pm)\alpha_2^\mp. \end{aligned} \tag{58}$$

7. The non-split $N=4$ SRSs

For brevity we confine ourselves to the non-split SRSs whose patching transition functions satisfy the additional 'chirality' constraints (which corresponds to the SU(2)-extended version of $N=4$ scf symmetry [29])

$$D_j^n\tilde{\theta}_i^{m_{ij}}=0 \tag{59}$$

where n, $m_{ij}=+,\ -$ and no summation here. Then H_{ij} becomes diagonal for $m_{ij}=n$ and off-diagonal for $m_{ij}=-n$, leading to $\mathrm{scf}_k\,H_{ij}=0$ (see (8)), which implies that H_{ij} are the $N=2$ scf matrices. So the superderivatives transform as

$$D_i^n=(D_i^n\tilde{\theta}_j^{-m_{ji}})\tilde{D}_j^{m_{ji}} \tag{60}$$

resulting in the definite 'chirality' transitions. Let

$$(m_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then the solution to the constraints (59) has the form

$$\begin{aligned} \tilde{\theta}_1^{n(ab)} &= \psi_1^n(Z^{ab}) + \theta_1^a g_{11}^{n,-a}(Z^{ab}) + \theta_2^b g_{12}^{n,-b}(Z^{ab}) + 2\theta_1^a \theta_2^b \lambda_1^n(z) \\ \tilde{\theta}_2^{n(cd)} &= \psi_2^n(Z^{cd}) + \theta_1^c g_{21}^{n,-c}(Z^{cd}) + \theta_2^d g_{22}^{n,-d}(Z^{cd}) + 2\theta_2^d \theta_1^c \lambda_2^n(z) \end{aligned} \tag{61}$$

where $Z^{ab} = z + \theta^a\theta^{-a} + \theta^b\theta^{-b}$. For all $m_{ij} = n$ the 'chirality' on the SRS is preserved and so it can be called a 'chiral' $N = 4$ SRS (e.g. see [29]). We shall consider this case only. It follows that the SCf conditions (3)-(5) lead to the matrix G (49) becoming the $N = 4$ scf matrix (see section 3). But now the additional restrictions on G differ from those of the split case (48) and read

$$G_{11}^{\mathrm{T}} G_{11}^{\mathrm{M}\prime} + G_{12}^{\mathrm{T}} G_{12}^{\mathrm{M}\prime} + G_{21}^{\mathrm{T}} G_{21}^{\mathrm{M}\prime} + G_{22}^{\mathrm{T}} G_{22}^{\mathrm{M}\prime} = G_{11}^{\mathrm{T}\prime} G_{11}^{\mathrm{M}} + G_{12}^{\mathrm{T}\prime} G_{12}^{\mathrm{M}} + G_{21}^{\mathrm{T}\prime} G_{21}^{\mathrm{M}} + G_{22}^{\mathrm{T}\prime} G_{22}^{\mathrm{M}} \tag{62}$$

$$G_{11}^{\mathrm{T}\prime} G_{12}^{\mathrm{M}\prime} + G_{21}^{\mathrm{T}\prime} G_{22}^{\mathrm{M}\prime} = 0. \tag{63}$$

The restriction for the odd functions is

$$\psi_1^{+\prime}(z)\psi_1^{-\prime}(z) + \psi_2^{+\prime}(z)\psi_2^{-\prime}(z) + \lambda_1^+(z)\lambda_1^-(z) + \lambda_2^+(z)\lambda_2^-(z) = 0 \tag{64}$$

having two inequivalent solutions

$$\lambda_1^\pm(z) = \psi_2^{\mp\prime}(z) \qquad \lambda_2^\pm(z) = \psi_1^{\mp\prime}(z) \tag{65}$$

and

$$\lambda_1^\pm(z) = \lambda_2^\mp(z) \qquad \psi_1^\pm(z) = \psi_2^\mp(z). \tag{66}$$

Consider first all G_{ij} to be the diagonal body $N = 2$ scf matrices. Then the SCf transformation will be the body one. Solving the SCf conditions (46), (47) and (62), (63) for this case and choosing the solution (65) we establish the general shape of the non-split 'chiral' $N = 4$ SRS transition functions in the following way:

$$\begin{aligned} \tilde{z} = f(z) &+ \hat{\theta}_1^+[\psi_1^-(Z^+)g_+(Z^+) - \psi_2^-(Z^+)h_-(Z^+)] \\ &+ \hat{\theta}_1^-[\psi_1^+(Z^-)g_-(Z^-) - \psi_2^+(Z^-)h_+(Z^-)] \\ &+ \hat{\theta}_2^+[\psi_2^-(Z^+)g_-(Z^+) + \psi_1^-(Z^+)h_+(Z^+)] \\ &+ \hat{\theta}_2^-[\psi_2^+(Z^-)g_+(Z^-) + \psi_1^+(Z^-)h_-(Z^-)] \\ &+ (\theta_1^+\theta_1^- + \theta_2^+\theta_2^-)[\psi_1^+(z)\psi_1^-(z) + \psi_2^+(z)\psi_2^-(z)]' \\ &- 2\hat{\theta}_1^+\hat{\theta}_2^+[\psi_1^-(z)\psi_2^-(z)]' - 2\hat{\theta}_1^-\hat{\theta}_2^-[\psi_1^+(z)\psi_2^+(z)]' + \theta_1^+\theta_1^-\theta_2^+\theta_2^- f''(z) \\ \tilde{\theta}_1^\pm &= \psi_1^\pm(Z^\pm) + \hat{\theta}_1^\pm g_\pm(Z^\pm) + \hat{\theta}_2^\pm h_\pm(Z^\pm) + 2\hat{\theta}_1^\pm\hat{\theta}_2^\pm\psi_2^{\mp\prime}(z) \\ \tilde{\theta}_2^\pm &= \psi_2^\pm(Z^\pm) - \hat{\theta}_1^\pm h_\mp(Z^\pm) + \hat{\theta}_2^\pm g_\mp(Z^\pm) + 2\hat{\theta}_2^\pm\hat{\theta}_1^\pm\psi_1^{\mp\prime}(z) \end{aligned} \tag{67}$$

where

$$g_+(z)g_-(z) + h_+(z)h_-(z) = f'(z) + \psi_i^+(z)\psi_i^{-\prime}(z) + \psi_i^-(z)\psi_i^{+\prime}(z) \tag{68}$$

and $Z^\pm = Z^{\pm\pm}$. Further, in line with various subgroups of (67), we can choose two of the G_{ij} as being monomial or vanishing, which leads to the disappearance of one term from the LHS of (68). If this is not the case, two local rotations (54) arise and are independent, as opposed to the split SRS, due to the absence of relation (55) between them now. Also, the non-split analogues of (56) and (58) can be constructed, as well as the case with $h(z)$ being pure soul. The $N=4$ extension of the fractional linear transformation $f(z)=(az+b)/(cz+d)$ follows from (67) after choosing $\psi_i^{\pm}(z)=(\gamma_i^{\pm}z+\delta_i^{\pm})/(cz+d)$. This is another way (cf [11]) to obtain the anomaly-free $N=4$ SCf transformations. Going to the infinitesimal form of (67) one can derive the explicit shape of the generators of $N=4$ SCf transformations, and the commutation relations between them define the centreless $N=4$ SCf algebra (see [1, 7] for more details).

Finally, we will have a look at the intriguing soul transformation (see section 5) involving no even functions at all:

$$\begin{aligned}\tilde{z} &= f(z)+(\theta_1^+\theta_1^-+\theta_2^+\theta_2^-)[\psi_1^+(z)\psi_1^-(z)+\psi_2^+(z)\psi_2^-(z)]' \\ &\quad +2\hat{\theta}_1^+\hat{\theta}_2^+[\lambda_1^+(z)\psi_1^-(z)-\lambda_2^+(z)\psi_2^-(z)] \\ &\quad +2\hat{\theta}_1^-\hat{\theta}_2^-[\lambda_1^-(z)\psi_1^+(z)-\lambda_2^-(z)\psi_2^+(z)]+\theta_1^+\theta_1^-\theta_2^+\theta_2^-f''(z) \qquad (69)\\ \tilde{\theta}_1^{\pm} &= \psi_1^{\pm}(Z^{\pm})+2\hat{\theta}_1^{\pm}\hat{\theta}_2^{\pm}\lambda_1^{\pm}(z) \\ \tilde{\theta}_2^{\pm} &= \psi_2^{\pm}(Z^{\pm})+2\hat{\theta}_2^{\pm}\hat{\theta}_1^{\pm}\lambda_2^{\pm}(z)\end{aligned}$$

where

$$f'(z)=\psi_i^{+\prime}(z)\psi_i^-(z)+\psi_i^{-\prime}(z)\psi_i^+(z) \qquad (70)$$

and (64) holds true. Note that for the fractional linear transformation (70) is solved by

$$f(z)=\frac{\delta_i^+\gamma_i^-+\delta_i^-\gamma_i^+}{c(cz+d)}+k \qquad (71)$$

and so it becomes degenerate after the body mapping. Moreover, it is not 'body preserving' [30] $\varepsilon(\tilde{z})\neq\varepsilon(z)$ and it has no infinitesimal form, hence there is no corresponding superderivation algebra [31]. The soul transformations (69) describe the transitions from the body to the soul and form a subsemigroup of the full $N=4$ SCf semigroup. Evidently, they should have non-trivial structure in soul directions (see [32]) and could be viewed as a 'bridge' between the pure body and soul worlds.

8. Conclusion

We have obtained the finite $N=4$ SCf transformations and classified them in terms of the permanent of the matrices having some specific properties. The latter have been studied as such and in connection with the plane non-Euclidean geometry. The manifest shape of the transformations derived allows some of them to be used to $N\leqslant 4$ split and non-split SRS transition functions and various SCf embeddings [33].

The careful analysis shows that the SCf symmetry has semigroup nature in general. The whole transformations viewed as solutions of the SCf conditions (3)–(5) without reference to invertibility form the $N=4$ SCf semigroup S_{scf} in fact. An element of S_{scf} is defined as a set of supersmooth even and odd functions determining an SCf transformation under consideration: thus, S_{scf} is an infinite-dimensional semigroup. In the case of the functions being fixed, S_{scf} is finitely generated and so the Cayley table characterizing S_{scf} entirely can be built in principle. Then an abstract semigroup corresponding to the semigroup of SCf transformations S_{scf} could be constructed. Furthermore, it should be noted that there is a subgroup $G_{scf} \subset S_{scf}$ (which is the ordinary SCF group) containing invertible elements and the unity (the identity transformation). The group G_{scf} is disjoint [21] because $S_{scf} = G_{scf} \cup I_{scf}$ and $G_{scf} \cap I_{scf} = \varnothing$, where I_{scf} is a proper ideal of S_{scf} (the set of the soul and zero transformations). The transformations from I_{scf} are partial ones having the degenerated body second projection and non-vanishing defect. Next, it would be exciting to consider soul foliations and similar features (e.g. along the lines of [17, 32]) from the semigroup viewpoint for refined building of a possible object which is close to SRSs but has transition functions from S_{scf} as a whole (rigorous consideration is a subject of separate study).

The present findings and constructions show that a thorough further investigation is needed to understand the essence and manifestation of extended SCF symmetry.

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